

Tau-functions and special solutions in a coupled Painlevé system*

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February 8, 2008

Abstract

For a pair of coupled Painlevé equations obtained as a scaling similarity reduction of the Hirota-Satsuma system we describe special parameter-families of solutions given in terms of mixtures of rational and Airy functions, and in terms of a second Painlevé transcendent. The tau-functions associated to some of these solutions are also given explicitly.

1 Introduction

In a recent pair of articles [4, 5] we have considered the following pair of coupled Painlevé equations,

$$\begin{aligned} L_1 L_1'' - \frac{1}{2}(L_1')^2 + (L_1 + 3L_2 + 2z)L_1^2 + \frac{1}{2}\ell_1^2 &= 0, \\ L_2 L_2'' - \frac{1}{2}(L_2')^2 + (3L_1 + L_2 + 2z)L_2^2 + \frac{1}{2}\ell_2^2 &= 0, \end{aligned} \tag{1.1}$$

which arise as a scaling similarity reduction of the Hirota-Satsuma system of partial differential equations. The system (1.1) is a coupling between two copies of the equation P34 in Ince's classification [7], to which it clearly degenerates via the consistent reductions $L_1 = 0 = \ell_1$ and $L_2 = 0 = \ell_2$. The similarity reduction was originally found in our thesis [6], but our interest in the pair of equations (1.1) was further stimulated by a conjecture that it is connected to a fifth order equation appearing in a classification of higher order Painlevé equations made by Cosgrove [2].

The Hirota-Satsuma system is itself a 4-reduction of the KP hierarchy [3], which means that the pair of equations (1.1) can be derived from an $sl(4)$ isomonodromic Lax pair. In our first article [4] we presented Bäcklund transformations (BTs) for the system (1.1), which could be interpreted in terms of a subgroup of the affine Weyl group $W(A_3)$ acting on the space of parameters (ℓ_1, ℓ_2) . In that article we also described families of special

*Isaac Newton Institute preprint NI02001-ITS.

lines in the (ℓ_1, ℓ_2) plane where the system admits two-parameter and three-parameter families of solutions.

The description of the BTs for the system (1.1) is much simplified by rewriting it as four coupled first order equations, namely

$$\begin{aligned} X'_1 &= -\frac{1}{2}L_1 - \frac{3}{2}L_2 - X_1^2 - z, \\ X'_2 &= -\frac{3}{2}L_1 - \frac{1}{2}L_2 - X_2^2 - z, \\ L'_1 &= 2L_1X_1 - \ell_1, \\ L'_2 &= 2L_2X_2 - \ell_2. \end{aligned} \tag{1.2}$$

Below we will present these BTs once again, and then show how they can be applied to obtain special families of solutions at isolated points in the parameter space. The second work [5] was primarily concerned with the tau-functions for the system (1.1) (or equivalently (1.2)), and the multilinear lattice equations connecting them. In what follows we give exact expressions for the tau-functions of some of the special solutions.

2 Bäcklund transformations

BTs for the coupled Painlevé equations (1.1) were first found in [4], but were presented more explicitly in terms of the variables of the system (1.2) in [5]. There are two basic reflections in the (ℓ_1, ℓ_2) plane:

$$R: (\ell_1, \ell_2) \rightarrow (\ell_2, \ell_1), \quad X_1 \leftrightarrow X_2, \quad L_1 \leftrightarrow L_2.$$

$$S: (\ell_1, \ell_2) \rightarrow (-\ell_1, \ell_2), \quad X_1 \rightarrow X_1^\dagger := X_1 - \frac{\ell_1}{L_1}, \quad X_2 \rightarrow X_2, \quad L_1 \rightarrow L_1, \quad L_2 \rightarrow L_2.$$

By combining the two reflectional symmetries R, S it is straightforward to obtain BTs connecting solutions of the system (1.2) at all the points

$$(\epsilon\ell_1, \epsilon'\ell_2), \quad (\epsilon\ell_2, \epsilon'\ell_1), \quad \epsilon, \epsilon' = \pm 1 \tag{2.1}$$

in the parameter space.

The affine symmetry of the solutions is generated by a translational BT, denoted T . For convenience we introduce the vector notation

$$\mathbf{l} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Then T is defined thus:

$$\begin{aligned} T: \quad \mathbf{l} &\rightarrow \mathbf{l} + \mathbf{c}, \quad X_1 \rightarrow \overline{X}_1 := X_2^\dagger - \frac{(\ell_1 + \ell_2 + 2)}{(L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z)}, \\ X_2 &\rightarrow \overline{X}_2 := X_1^\dagger - \frac{(\ell_1 + \ell_2 + 2)}{(L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z)}, \end{aligned} \tag{2.2}$$

$$\begin{aligned}
L_1 &\rightarrow \overline{L}_1 := L_2 + \frac{\ell_1 + \ell_2 + 2}{L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z} \left(2X_1^\dagger - \frac{(\ell_1 + \ell_2 + 2)}{(L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z)} \right), \\
L_2 &\rightarrow \overline{L}_2 := L_1 + \frac{\ell_1 + \ell_2 + 2}{L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z} \left(2X_2^\dagger - \frac{(\ell_1 + \ell_2 + 2)}{(L_1 + L_2 + 2X_1^\dagger X_2^\dagger + 2z)} \right).
\end{aligned}$$

This transformation is explicitly invertible; the exact expression for T^{-1} may be found in [5].

3 Hamiltonians and tau-functions

Part of our original motivation for deriving the system (1.1) was that it may be written as a non-autonomous version of an integrable Hamiltonian system of two particles interacting via a quartic potential. More precisely, (1.1) arises from the Hamiltonian

$$h = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{8}(q_1^4 + 6q_1^2 q_2^2 + q_2^4) + \frac{1}{2}z(q_1^2 + q_2^2) - \frac{1}{8} \left(\frac{\ell_1^2}{q_1^2} + \frac{\ell_2^2}{q_2^2} \right) \quad (3.1)$$

by making the transformation to canonical conjugate coordinates and momenta according to

$$L_j = q_j^2, \quad L'_j = 2p_j q_j, \quad j = 1, 2.$$

It turns out that this system is related by a canonical (contact) transformation to another Hamiltonian with quartic potential, namely

$$H = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{16}(Q_1^4 + 6Q_1^2 Q_2^2 + 8Q_2^4) + \frac{1}{2}z(Q_1^2 + 4Q_2^2) - \frac{2\nu^2}{Q_1^2} - \xi Q_2, \quad (3.2)$$

with the parameters related by

$$\mathbf{m}[\mathbf{l}] := \begin{pmatrix} \nu \\ \xi \end{pmatrix} = \begin{pmatrix} \ell_1 + \ell_2 - 2 \\ \ell_1 - \ell_2 \end{pmatrix}.$$

The autonomous versions of the Hamiltonians (3.1,3.2) were considered in [1].

Hamilton's equations for (3.2) lead to another coupled Painlevé system, which is a coupling between P34 and the second Painlevé equation P2. Indeed, simple reflectional symmetries of this other system lead to a derivation of the translational BT T for the original system (1.1) (see [4, 5] for further details). Here we simply wish to note that the Hamiltonians (3.1,3.2) can be expressed as logarithmic derivatives of holomorphic tau-functions for the system. This is the analogue in this higher order setting of Okamoto's results [8, 9] on the tau-functions and Hamiltonian structures of the Painlevé equations. Associated to the first Hamiltonian (3.1) is a tau-function, denoted τ in [5], such that

$$h_1(z) = \frac{d}{dz} \log \tau(z) \quad (3.3)$$

(the subscripts signify the dependence on the parameters). Corresponding to the second Hamiltonian (3.2) related by a canonical transformation there is another tau-function, denoted ρ , with

$$H_{\mathbf{m}[\mathbf{l}]}(z) = -2 \frac{d}{dz} \log \rho_{\mathbf{m}[\mathbf{l}]}(z). \quad (3.4)$$

In terms of the variables in the coupled system (1.2), these two Hamiltonians are related by the formula

$$H_{\mathbf{m}[1]} = -4h_1 - 2(X_1 + X_2).$$

By making use of the BTs we derived bilinear and multilinear lattice equations for the tau-functions at different points in parameter space, and expressed the variables L_j, X_j explicitly in terms of them. The reader is referred to [5] for the exact formulae.

4 Special lines and special points

We expect that the solutions of the system (1.1) define new transcendental functions at generic points in parameter space, and this expectation is supported by work of Cosgrove [2]. However, in [4] we showed that along three families of lines (denoted \mathcal{L}_j) in the (ℓ_1, ℓ_2) plane the system admits parameter families of solutions in terms of known (P2, or equivalently P34) transcendents.

- \mathcal{L}_1 : Along the lines

$$\ell_1 = 2n \quad \text{and} \quad \ell_2 = 2n, \quad n \in \mathbb{Z},$$

there is a three-parameter family of special solutions to the system (1.2), obtained by applying the BTs R, S and T to the reduction $L_2 = 0$ on the line $\ell_2 = 0$, with L_1 satisfying P34.

- \mathcal{L}_2 : On the lines

$$\ell_1 \pm \ell_2 = 4n, \quad n \in \mathbb{Z},$$

there is a two-parameter family of special solutions, obtained by application of the BTs to the reduction $L_1 = L_2 = L$ on the line $\ell_1 - \ell_2 = 0$, where L satisfies P34 with a different scaling to the \mathcal{L}_1 case above.

- \mathcal{L}_3 : On the lines

$$\ell_1 \pm \ell_2 = 2(2n + 1), \quad n \in \mathbb{Z},$$

there is a three-parameter family of special solutions, generated by application of the BTs to a special reduction on the line $\ell_1 + \ell_2 = 2$, which corresponds to the reduction $Q_1 = 0 = \nu$ in the second Hamiltonian system (3.2), with Q_2 satisfying P2.

In [5] we described how particular rational solutions could be found at the points

$$(\ell_1, \ell_2) = (2(m + n) + \epsilon/2, 2(m - n) + \epsilon'/2), \quad \epsilon, \epsilon' = \pm 1, \quad (m, n) \in \mathbb{Z}^2$$

on the lines \mathcal{L}_2 . In [4] we mentioned that solutions in terms of mixtures of rational and Airy functions could be obtained at the intersections of the families \mathcal{L}_j . Below we consider these intersections more explicitly.

4.1 Intersection of \mathcal{L}_1 and \mathcal{L}_2

A result originally appearing in our thesis [6] was that at the point $(0,0)$ in parameter space a separation of variables is possible and the general (four-parameter) solution of the system (1.1) is given in terms of two copies of P2 with zero parameter. More precisely, the variables in the system (1.2) are given by

$$L_1 = -(y_+ + y_-)^2, \quad L_2 = -(y_+ - y_-)^2, \quad X_1 = (\log(y_+ + y_-))', \quad X_2 = (\log(y_+ - y_-))',$$

where y_{\pm} are two solutions of

$$y'' - 2y^3 + zy = 0. \quad (4.5)$$

In terms of P2 tau-functions T, \bar{T} related by a Bäcklund transformation, the solution of (4.5) is expressed as

$$y = (\log \bar{T}/T)'$$

Then it turns out that the tau-function τ for the system (1.2) at point $\mathbf{0} = (0,0)^T$ in parameter space is given by a product of four P2 tau-functions,

$$\tau_{\mathbf{0}} = T_+ T_- \bar{T}_+ \bar{T}_-,$$

while the other tau-function ρ is given by

$$\rho_{\mathbf{0}} = (y_+^2 - y_-^2) \tau_{\mathbf{0}}^2.$$

By applying the BTs R, S, T the general solution of the system is obtained at all the intersection points of the lines $\mathcal{L}_1, \mathcal{L}_2$ in terms of two solutions y_{\pm} of P2.

4.2 Intersection of \mathcal{L}_1 and \mathcal{L}_3

A one-parameter family of mixed rational-Airy solutions may be generated at these points by applying the BTs to the seed solution

$$L_1 = -2z, \quad L_2 = 0 = X_1, \quad X_2 = Y, \quad Y' + Y^2 = 2z$$

at the point $\mathbf{p} = (2,0)^T$. For example, applying the translational BT T yields a corresponding one-parameter solution at the point $\mathbf{p} + \mathbf{c} = (4,2)^T$,

$$L_1 = \frac{4}{Y} - \frac{4z^2}{Y^2}, \quad L_2 = 2z - \frac{4z^2}{Y^2}, \quad X_1 = Y - \frac{2z}{Y}, \quad X_2 = \frac{1}{z} - 2zY.$$

The free parameter comes from the solution of the Riccati equation for Y , which is linearized to Airy's equation (with suitable scaling), viz

$$Y = (\log \phi)', \quad \phi'' - 2z\phi = 0.$$

The associated tau-functions are

$$\tau_{\mathbf{p}} = \exp(-z^3/6), \quad \rho_{\mathbf{p}} = \phi \exp(-z^3/3), \tau_{\mathbf{p}+\mathbf{c}} = \phi' \exp(-z^3/6), \quad \rho_{\mathbf{p}+\mathbf{c}} = \phi z \exp(-z^3/3).$$

All solutions obtained from this by application of the BTs are rational functions of z and Y .

4.3 Intersection of \mathcal{L}_2 and \mathcal{L}_3

Another one-parameter family of mixed rational-Airy solutions is found at the $\mathcal{L}_2, \mathcal{L}_3$ intersection points starting from the seed solution

$$L_1 = L_2 = -J^2 - z, \quad X_1 = X_2 = -J, \quad J' + J^2 = -z$$

at the point $\mathbf{q} = (1, 1)^T$. So for instance, at $\mathbf{r} = TRSR \cdot \mathbf{q} = (3, 1)^T$ we find the solution

$$L_1 = -z - \frac{2}{J} - \frac{z^2}{J^2}, \quad L_2 = -z - \frac{z^2}{J^2}, \quad X_1 = \frac{z}{J}, \quad X_2 = \frac{z}{J} + \frac{1}{J^2 + z}.$$

The Riccati equation for J is linearized to a rescaled Airy's equation,

$$J = (\log \psi)', \quad \psi'' + z\psi = 0.$$

The corresponding tau-functions at these points are

$$\tau_{\mathbf{q}} = \psi, \quad \rho_{\mathbf{q}} = 1, \quad \tau_{\mathbf{r}} = \psi', \quad \rho_{\mathbf{r}} = \begin{vmatrix} \psi & \psi' \\ \psi' & \psi'' \end{vmatrix}.$$

5 Acknowledgements

This poster was presented at the NEEDS conference at the Isaac Newton Institute, 25th-31st July 2001. I would like to thank the INI for supporting my stay in Cambridge during the Programme on Integrable Systems.

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